H.A. Lorenz, drawing (1913) by Willy Sluiter (Photograph Iconografisch Bureau, The Hague)

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H.A. Lorentz: Sketches of his work on slow viscous flow and some other areas in fluid mechanics and the background against which it arose.

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1. Introduction

With this special issue of the Journal of Engineering Mathematics we commemorate and celebrate the appearance, one hundred years ago (Fig. 1), of a paper [1] by the Dutch physicist H.A. Lorentz in which he put forward some seminal ideas on slow viscous flow (see also [2-4]). Lorentz (to be pronounced as Lawrence with emphasis on the first syllable) is not known, per se, for his contributions to fluid mechanics. Indeed, he was a physicist whose fame rested first and foremost on his contributions to the theory of electromagnetism, electrodynamics, the theory of electrons and the dawn of relativity. His place among his contemporaries was, perhaps, described best by Albert Einstein who wrote ([5] and [6, pp73-76]) in 1953:

"At the turn of the century the theoretical physicists of all nations considered H.A. Lorentz as the leading mind among them, and rightly so." But then, Einstein continues as follows:

"The physicists of our time are mostly not fully aware of the decisive part which H.A. Lorentz played in shaping the fundamental ideas in theoretical physics. The reason for this strange fact is that Lorentz's basic ideas have become so much a part of them that they are hardly able to realize quite how daring these ideas have been and to what extent they have simplified the foundations of physics."

Einstein is here referring to the theory of electrons which explains the electric and magnetic properties of matter in terms of charge and motion of charged atomic particles - the electrons - and which has led to concepts now known to us as the Lorentz transformation, the Lorentz-Fitzgerald contraction, the Lorentz force, Lorentz invariance, the Lorentz condition and many, many more.

Curiously enough, the ideas he put forward in [1] have met with a similar fate, at least to some extent. Although contemporaries, *e.g.* [7], fully recognise the importance of his work in fluid mechanics, later works, particularly papers appearing in the scientific literature, often refer to other, much later [8] sources for some of these ideas. In this context it must be recorded that the most excellent survey on creeping flows by Happel and Brenner [9] does emphasize the importance of the Lorentz reciprocal theorem, the Lorentz integral-equation formulation and the Lorentz reflection theorem, which were all put forward with extreme clarity in [1]; such a seminal paper, only six or seven journal pages long and containing at least three important ideas!

The question arises why Lorentz, who was not a professed fluid dynamicist, should have decided to tackle some problems in that particular field. Indeed, apart from [1], and appearing at about the same time, there is a paper on turbulent flow in pipes [10] and a few other contributions. That he gave extended renditions in German of both [1] and [10] in 1907 [11]

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gut möglich ist, mittels des Suspensionsverfahrens selbst bei ziemlich kleinen Fröschen und bei erhaltener Circulation, die Bewegungen von Venen, Sinus, Atrien und Ventrikel gleichzeitig messbar zu registriren, wird die technische Lösung der zahlreichen und verwickelten Probleme, die hier in Angriff genommen werden müssen, voraussichtlich nicht von unüberwindlicher Schwierigkeit sein. Ueber einige auf diesem Wege bereits erhaltene Resultate hoffe ich der K. Akademie in einer der nächsten Sitzungen Mittheilungen machen zu können.

Natuurkunde. — De Heer Lorentz biedt eene mededeeling aan, getiteld: „*Eene algemeene stelling omtrent de beweging eener vloeistof met wrijving en eenige daaruit afgeleide gevolgen*".

§ 1. De bewegingsvergelijkingen voor eene onsamendrukbare vloeistof met wrijving kunnen geschreven worden in den vorm

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\left.\begin{aligned}
\varrho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) &= \quad + \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right)\\
\varrho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) &= \mathbf{Y} + \left(\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\right)\\
\varrho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) &= \mathbf{Z} + \left(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\right),
\end{aligned}\right\} \quad . \quad (2)$$

Daarin stellen voor:
ϱ de dichtheid,
u, v, w de snelheidscomponenten,
X, **Y**, **Z** de componenten der uitwendige kracht per volumeenheid,
X_x, X_y, enz. de spanningscomponenten op vlakken, loodrecht op de coordinaatassen.

De waarde dezer laatste grootheden wordt, als p den druk en μ den wrijvingscoefficient voorstelt, bepaald door de vergelijkingen

$$X_x = -p + 2\mu\frac{\partial u}{\partial x}, \text{ enz.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$X_y = Y_x = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \text{ enz.} \quad . \quad . \quad . \quad . \quad (4)$$

Fig. 1. First page of ref. [1], complete with type-setter's error.

indicates that he considered these two the most important of his works in fluid mechanics, that is, at that particular time. Since I do not know of any written evidence explaining why Lorentz embarked on those studies, nor does Dr. Kox [12], who is in the process of collating and publishing Lorentz's extensive correspondence, we must guess at the reasons for his having done so.

During his entire life as a cognizant scientist, Lorentz had always been a strong advocate of the *aether* concept. Even in his later years, when modern developments in physics showed that this concept was no longer needed, he refused to fully divest himself of it. During its heyday, the aether was considered to be a medium with rather ill-defined, but possibly fluid-like properties.

An important problem was the interaction of heavenly bodies, such as the earth, with a continual aether flow of a kind which was thought to be emanating from the sun. Whether the aether behaved as a turbulent or a creeping flow or as a solid body, or alternatively was just an immobile medium, was still a moot question. Earlier, another famous name in slow viscous flow, Stokes [13], had tried to come to grips with the nature of the aether. It was Kelvin [14] who defined the aether as an incompressible fluid in turbulent motion. It is easily imaginable that, in the scientific climate of the day, Lorentz decided to indulge in some fluid mechanics himself.

At that time, one of the hot topics in physics was the famous experiment of Michelson and Morley, which aimed at establishing the influence of the earth's motion on the velocity of light, *i.e.* aether motion. The negative outcome of that experiment held the physicist of those days in its grip. In a series of lectures on aether theories and aether models delivered in Leiden (Leyden) during 1901-1902 [15], Lorentz attempted to understand the nature of the aether on the basis of various fluid-like models. That particular study culminated in theories for the motion of a single or several spheres through an inviscid fluid. This makes it easy to understand, and the more so because the subject matter was still very undecided, why he digressed to study the motion of bodies through highly viscous liquids.

In what follows I shall give a précis of Lorentz's viscous-fluid paper [1], commenting on its contents as I go along. Next, I shall describe his other great venture in fluid mechanics, namely his extensive theoretical modelling on the so-called Zuiderzee project [16]. Although the latter study concerns flows in which viscosity plays a minor role only, it is interesting enough to be reported here, not in the least because it is a veritable piece of engineering mathematics. Paraphrasing Einstein's words, we could say that this work is not as widely known as it deserves to be. A concise report on its contents may be of interest to some of our readers.

Again, one should realise that Lorentz's fluid-mechanics work is only a very small part within his general œuvre. His nine-volume set of collected works [17] mentions only a few papers in fluid mechanics in Vol. IV. There is some further fluid mechanics in Vol. VII. The Zuiderzee project is not mentioned at all. There is some more work on the motion of gases, but most of it is in relation to kinetic theories.

Even so, limited in size as it may seem to be in relation to all his other work, Lorentz's fluid-mechanics work is by no means of passing interest only. This is what we have set out to demonstrate with this special issue.

2. Lorentz's fluid-mechanics paper of 1896

The paper begins with the formulation of the basic equations governing the flow of an incompressible viscous fluid with body forces. Lorentz considers a closed surface σ within the fluid, thus enclosing a given volume. Within this volume he distinguishes two different states of motion. In a succession of very clear arguments he is then able to derive a complicated integral relationship which relates the surface stresses to the inertial and body forces in the enclosed volume. This is his theorem I. From then onwards he restricts himself to flows for which the Reynolds number vanishes - in his terminology this means that the velocities are infinitesimal everywhere - and he disregards the influence of body forces. His complicated theorem I is then reduced to an extremely clear and useful theorem II which is now called the Lorentz reciprocal theorem [9]:

$$\int (u'X_n + v'Y_n + w'Z_n)\mathrm{d}\sigma - \int (uX_n' + vY_n' + wZ_n')\mathrm{d}\sigma = 0, \tag{1}$$

where u, v, w are the velocity components, X_n, Y_n, Z_n are the components of the viscous force acting at the surface and a prime distinguishes the second state of motion from the first.

In the remainder of the paper he exploits this theorem to obtain two further and equally important results. First, he writes down the equations describing the flow due to a sphere of radius R moving at a velocity c, which he defines as infinitesimally small, through a highly viscous fluid. This solution was derived by Stokes [18]. From this solution Lorentz derives a simpler-looking but very meaningful singular solution, which is now known as the *stokeslet*, although it was in this paper by Lorentz [1] that this solution appeared for the first time (see below). However, he brushes over the way in which he derived this singular solution, which he does not present explicitly. It is only by studying his Eq. (7) that we may understand that he really derived the stokeslet

$$u = \frac{3}{4}\left(\frac{x^2}{r^3} + \frac{1}{r}\right)\xi, \quad v = \frac{3}{4}\frac{xy}{r^3}\xi, \quad w = \frac{3}{4}\frac{xz}{r^3}\xi, \tag{2}$$

where $\xi = Rc$. All he says is that R tends to zero in the Stokes solution. Lorentz, the physicist, must have felt uncomfortable with the fact that the velocity of his second state of motion became infinite at $r = 0$, even if ξ was infinitesimally small but fixed, since that would violate his earlier requirement that the velocity should be infinitesimal everywhere. This might explain why he did not present Eq. (2) explicitly in [1]. Writing down the Reynolds number $|(u, v, w)|r/\nu$, where ν is the kinematic viscosity, we see that this tends to zero everywhere when $\xi \to 0$, which explains why (2) is a perfectly admissible creeping-flow solution. It would seem, therefore, that he need not have been as concerned about this as he may have been. Anyway, going from the perfectly physical Stokes solution to the more elusive singular solution is a step which, in Einstein's terminology, is daring. In our days, many of us are familiar with the idea of a double limit leading to an interesting singularity, but in Lorentz's days this was a daring step, hesitant and partly obscured as it may seem to have been taken. Lorentz must have felt intuitively that the major result he was about to derive, I mean the integral-equation formulation, justified his taking this step.

Consulting the more extended German [2] version of [1], we discover that, ten years later, Lorentz wrote down Eq. (2) without any reservations with $\xi = K/6\pi\mu$, where K is a force and μ the dynamic viscosity. Apparently, it was Hancock [19] who invented the term stokeslet.

He observed that the Stokes solution for the flow induced by a sphere moving through a highly viscous fluid can be regarded as the sum of two solutions which are singular at the centre of the sphere. One of these is a doublet which induces an irrotational flow. Hancock then continues and writes: "... and the other is a singularity peculiar to viscous motion, which will here (for want of a better word) be called a *Stokeslet*...". This apologetic phraseology indicates that he was somewhat uncertain when he wrote it down. We now know that he might have coined a more appropriate term: the *lorentzlet*; Year of birth: 1896.

Using the well-known device of surrounding the point where the force acts by a small sphere, making this part of σ and then allowing its radius to vanish, Lorentz derived the famous integral-equation formulation for slow viscous flows which relates the velocity vector at any point inside the fluid to a boundary integral which involves the stresses and the velocities on σ. Ever since, and particularly during the past two decades, this formula has been used extensively in so-called boundary-element calculations of slow viscous flows. In numerical calculations this formulation turns out to be far superior to the familiar formulation based on the vorticity and the stream function. Despite its fame and importance, it is not always recognised that it was Lorentz who derived this rule a century ago.

Another very nice application of the reciprocal theorem is the Lorentz reflection formula. Lorentz asked himself the following question. Suppose we know a flow in full three-dimensional space, *i.e.* we know the velocities and the pressure at every point in R_3. Let us now introduce a plane surface dividing infinite space into two semi-infinite halves. Can we now express the velocity and the pressure in terms of the original solution? Lorentz tells us that it can be done when we use his reciprocal theorem and he shows the result. It is amusing to read that he leaves out the various mathematical steps needed to achieve this, since this would consume too much journal space! In the later German version of the paper [2, page 38], he tell us how he did it.

Professor J.B. Keller [20] has pointed out that the Eqs. (9) of [1] can yield surprising results. He considers a shear flow with the following velocity components and pressure: $u_1 = Uy$, $v_1 = w_1 = p_1 = 0$. This flow does not decay at infinity but it does satisfy the full Navier-Stokes equations. Then Lorentz's Eqs. (9) yield $u_2 = Uy$, $v_2 = -2Ux$, $w_2 = 0$, $p_2 = 0$. This solution is surprising because it has a nonvanishing component v_2. Furthermore this solution is not unique, since for arbitrary U, V, W, $u_2 = Uy$, $v_2 = Vx$, $w_2 = Wx$, $p_2 = 0$ is a Stokes flow which also satisfies Lorentz's Eq. (8). The explanation is that Lorentz's result (9) was derived under the unstated assumption that certain integrals over a large sphere vanish as the sphere radius tends to infinity. Only then is there a unique solution. For the flows above, this assumption is violated. The fact that Lorentz made this assumption is clear from the derivation in [2], but it is not mentioned in [1].

The paper concludes with a discussion of flows within a region bounded by an arbitrary surface σ and Lorentz derives a few useful results for these.

3. The Zuiderzee project

For most of his conscious life Lorentz had been a theoretical physicist mainly interested in understanding the electromagnetic field and its interaction with charged particles. But then, at the age of sixty five, he became a mathematical engineer and oceanographer! In 1918 the Government asked him to take up the presidency of a committee whose task it was to investigate the effect of a proposed giant dam on coastal sea-water levels during gale conditions. This dam was to be the *pièce de résistance* of the biggest engineering project ever to be undertaken in the country: the closing off of the Zuiderzee (Zuyderzee), a relatively large inland body of water. This project aimed at achieving two goals: 1) to protect forever the inland coast from the ravages of big storms which had caused devastating inundations in the past and 2) to reclaim land for agriculture and human habitation. The geography of the pertinent part of The Netherlands is shown in Fig. 2.

Eight years later the committee presented their final report [16] which contained a wealth of information. On the basis of its findings, clear-cut decisions could be made on how to make the dam. Its construction was completed in 1932. During subsequent storms the predictions of the report have proved their great worth. Most of these predictions resulted from theoretical models which were formulated, analysed and computed by Lorentz himself. Later reports [21] confirm that it was he, and he alone, who carried the modelling burden, because no one else really understood how these things worked. In what follows I shall try to give the reader an idea of Lorentz's approach.

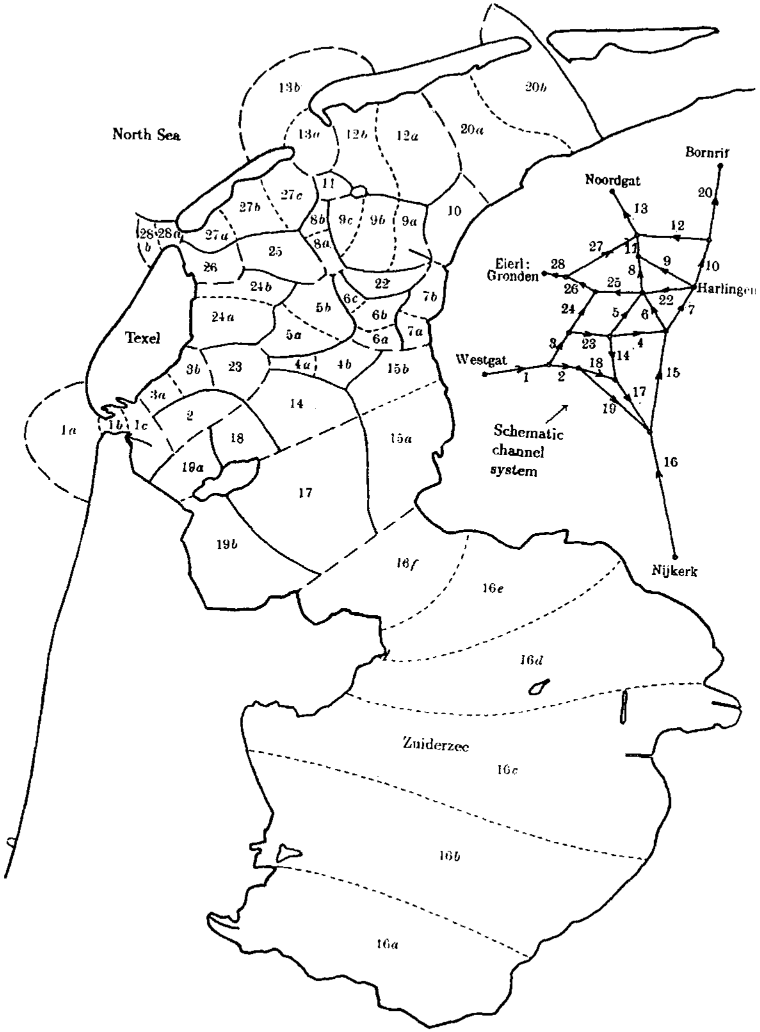
Reading through [16], as I did for the purpose of writing this essay, one is struck with admiration for the way Lorentz proceeded to tackle an incredibly complicated problem. At a time when the only numerical tools were a slide rule and some awkward mechanical calculation machines, he set out to come to grips with the problem of wave propagation and flow in a shallow-water area of complicated shape and produce results which should be useful from a practical point of view.

The basic idea was simple enough: Construct a model which predicts water levels and flow diagrams, then ascertain its accuracy by comparing its predictions with known observations from storms which had occurred in the past and then apply the method to the new geometry which will exist when the proposed dam will have been put in its place (Fig. 3). In the report three models were put forward, the first two resulting from existing practices in oceanography. Lorentz, however, proposed a third method which was based on the exact equations of fluid motion, which the other two weren't, and then reduce its complexity and make sensible approximations, so that calculations became feasible. It is clear from the report that the third method got the upper hand.

Lorentz's approach ran as follows. Realising that it was impossible to make a three or even two-dimensional model for a shallow sea with a bottom topography showing depths ranging from zero to thirty metres, with rapid depth variations occurring in many places, he decided to model the inland sea as a system of interconnecting channels (Fig. 4). To account for depth variations in a sideways direction, each of these single channels consisted of a series of parallel channels. In Fig. 4, the line bc represents the location of the proposed dam. Once it had been constructed, the channels bv and cv would have been cut off. At the points a, e, g, k and m, which represent the inlets between the islands, known observations of water levels could be used. These were measured during historical storms. At v the water flow into the Zuiderzee, as it was measured during those same storms, was entered into the problem as a boundary condition. At l there is another outflow into a much smaller inland sea.

At first, Lorentz assumed the flow situation to be stationary, *i.e.* water levels at the inlets and the outflow at v did not vary in time. Of course, he realised the shortcomings of this assumption, but the complication of the model required this restraint. Then, if l measures distance along a channel and h is the elevation of the surface of the sea above equilibrium, the following equation is appropriate

$$\frac{\mathrm{d}h}{\mathrm{d}l} - \frac{F\cos\vartheta}{g\rho q} + \frac{1}{C^2 q}|v|v = 0\,. \tag{3}$$



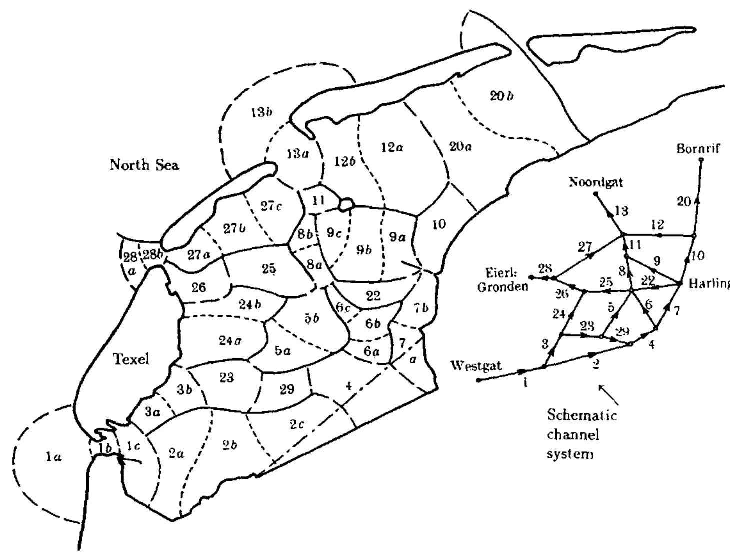


Fig 2. Geography of the central and North-Western parts of The Netherlands depicting the situation of the early twentieth century when the Zuiderzee was still connected with the North-Sea. The numbers 1*a* – 28 signify the channel system of Lorentz's most extended model. The theoretical network used in Lorentz's calculations is shown to the upper right. (From ref. [16], Fig. 29, page 100).

Fig 3. The channel system for the situation after the closing dam would have been constructed. (From ref. [16], Fig. 30, page 101).

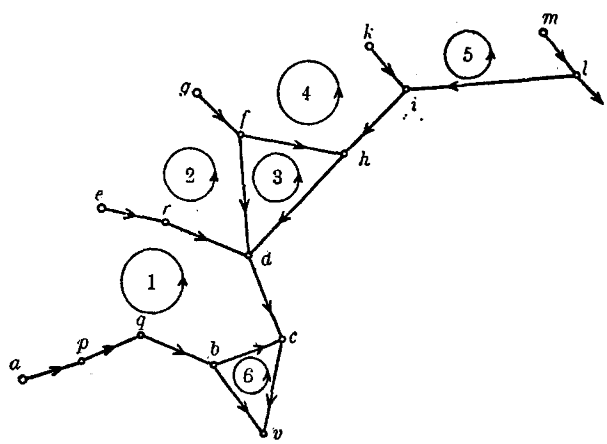


Fig. 4. Simple channel network showing six loops as used by Lorentz in a pilot calculation. (From ref. [16], Fig. 37, page 161).

Here F is the wind stress acting at the surface, but averaged over the depth of the channel, q is the equilibrium depth and v is the (average) flow velocity. Further, ϑ is the angle between the direction of l increasing and that of the wind, g is the acceleration due to gravity and ρ the density of water. Finally, C is a constant representing friction between the water and the sea-bed. It is called Eytelwein's constant and its value is about 50 $\mathrm{m}^{1/2}/\mathrm{s}$. The term $|v|v$ expresses that bottom friction forces always act against the direction of flow and are quadratic.

Integrating Eq.(3) from one end (P) of the channel to the next (Q), Lorentz obtained

$$h_{\mathrm{Q}} - h_{\mathrm{P}} = \frac{l}{g\rho q} F_l - \frac{l}{b^2 q^3 C^2} |s| s \,, \tag{4}$$

assuming that conditions and parameters remained the same along the entire length of the channel. If this were too unrealistic an assumption, then a channel was broken up into two or more channels. In Eq.(4), b is the width of the channel and $s = bqv$ is the total flow through it.

Lorentz's calculation procedure ran as follows. In a pilot calculation he used values that were recorded during the well-documented storm of 22/23 December 1894. He then considered a series of open and closed loops as shown in Fig. 4 which he numbered ① – ⑥. Starting with loop ① he knew h_a (h_{P} in Eq. (4)), but s was unknown along the series of channels $a \rightarrow b$. So, before being able to continue, he had to assume a value for this unknown. Since all the other values in Eq.(4) were known from $a \rightarrow b$, he was able to find provisional values of h_p, h_q and h_b. Then he arrived at loop ⑥, where the efflux s into the Zuiderzee was known (210000 m^3/s). Working his way through the loop system and working away on his slide rule, until he reached the end at the points m and l, he had to assume the values of a few other unknowns. Using an iteration procedure, he then refined these estimates and arrived at the final result.

Table 1. Channel dimensions for the network as shown in Fig. 4. Channel lenghts b_i in km, widths q_i in metres. The (approximate) angle between wind direction and channel direction is given by ϑ for the storm of 22/23 Dec. 1894. (From [16], Table 19, page 162)

Channel	Length	Channel cross profile			
		Main channel	Subchannel 1	Subchannel 2	
	(km)	$b_1 \times q_1$	$b_2 \times q_2$	$b_3 \times q_3$	ϑ
ap	10	8.0 × 9.0	2.0 ×20.0	–	70°
pq	13	2.0 ×27.0	2.0 ×15.0	2.0 × 8.0	70°
qb	15	20.5 × 4.0	1.5 ×20.0	1.0 ×10.0	0°
bv	15	17.5 × 6.25	–	–	0°
bc	16	12.0 × 6.0	–	–	65°
cv	15	14.8 × 7.0	–	–	55°
dc	17	9.0 × 6.0	1.5 ×16.0	10.5 × 4.0	30°
er	8.5	1.5 ×10.0	2.5 × 6.0	4.0 × 1.0	30°
rd	12	14.0 × 3.5	2.0 × 8.0	–	30°
fd	19	1.3 ×18.0	9.7 × 4.0	2.0 × 9.0	20°
gf	9	11.0 × 9.0	2.0 ×18.0	–	0°
fh	14	1.0 ×16.0	2.0 × 9.0	5.0 × 4.0	40°
hd	16	4.0 × 6.5	6.0 × 3.5	–	90°
ih	13	16.0 × 5.0	–	–	105°
ki	14	1.2 ×21.0	3.8 × 5.0	–	0°
li	19	8.5 × 5.5	–	–	120°
ml	13	11.0 × 5.0	1.5 ×12.0	–	0°

To give an idea of the dimensions and parameter values of this calculation, some input values are listed in Table 1. The end result is shown in Fig. 5, where it is understood that the values of h at a, e, g, k and m and the values of s at v and l are known from observation. It turned out that the calculated values of h at the other points agreed very well with the observed ones.

Next, the same calculation was carried out for the closed-off situation, where bc is now alongside the closing dam. The final result is shown in Fig. 6. As far as the flow pattern is concerned, there is a striking difference with that of Fig. 5. Whereas in the open situation water flowed in through all the inlets, the closed situation shows that inflow occurs through four inlets, but now there is an outflow through the inlet between the island of Texel and the mainland. That something like this should happen is understandable when it is realised that the storm of 1894 was a north-westerly one. The flow levels are much lower, but the surface elevation is higher by more than a metre at the location of the proposed dam.

Later, Lorentz carried out more refined calculations, using a system of channels as depicted in Fig. 2. Table 6 of [16] reveals that the total number of channels and sub-channels in the calculation was 120. The iteration procedure described above must indeed have been a very laborious undertaking which he shared with his assistant Thijsse [21]. Apart from a slide rule, they used one of the early multiplication machines called Millionär. This machine is described by Galle [22, pp39-44] and makes an awkward impression, at least by our standards. Indeed, its only advantage over the slide rule was accuracy (eight decimals), not speed of calculation.

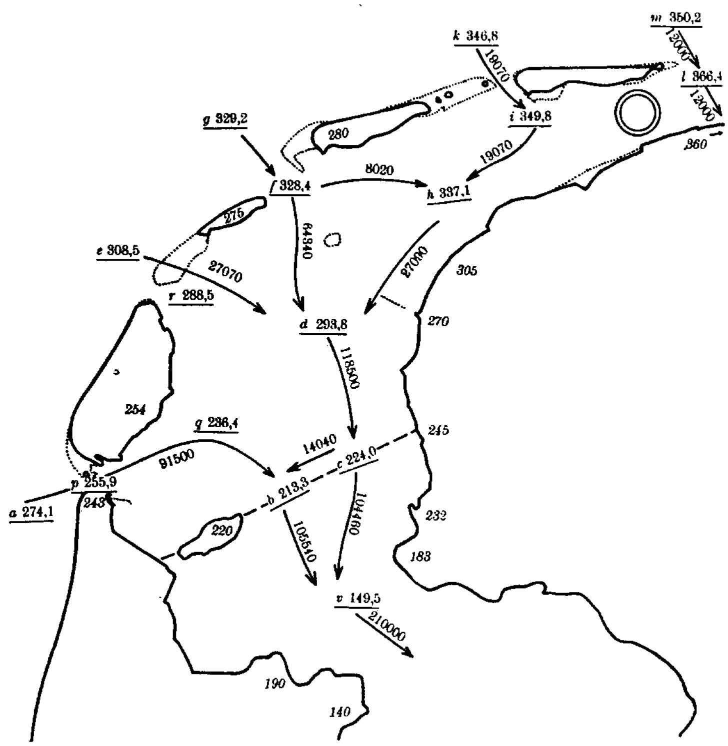


Fig. 5. Flow directions and levels (m^3/s) and sea level rises (cm) as calculated by Lorentz for the storm of 22/23 Dec 1894 using the channel network of Fig. 4. (From ref. [16], Fig. 38, page 166).

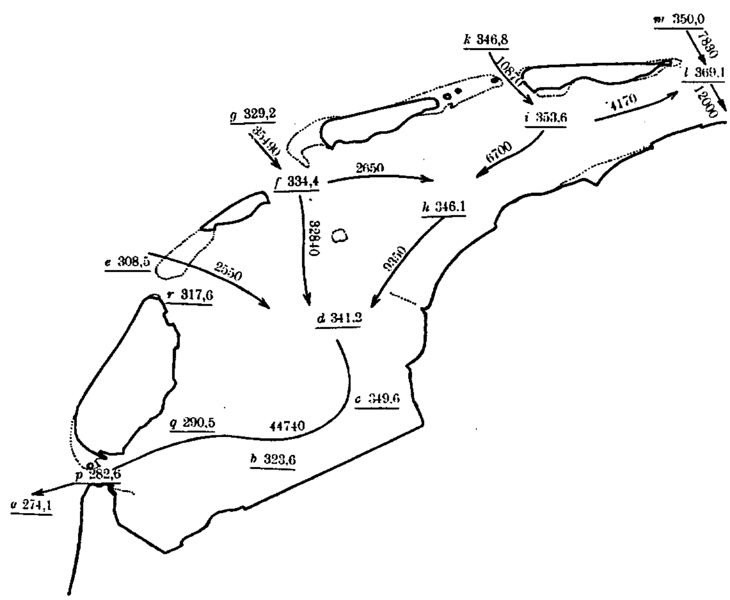


Fig. 6. Fictitious flow directions and levels (m^3/s) and sea level rises (cm) as calculated by Lorentz for the storm of 22/23 Dec 1894, assuming that the closing dam would already have been constructed and using the channel network of Fig. 4. (From ref. [16], Fig. 39, page 169).

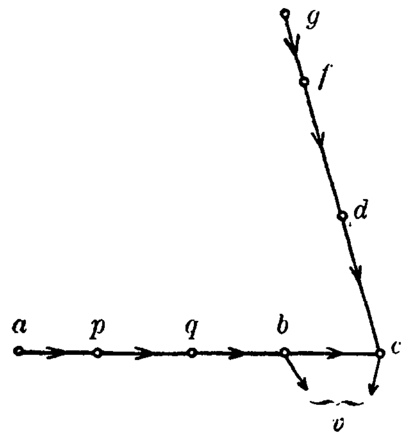


Fig. 7. Simple channel system as used by Lorentz in a time-dependent calculation. (From ref. [16], Fig. 45, page 179).

Realising that the assumption of stationary conditions may not be sufficiently accurate for the modelling of the occurrences during a violent storm, Lorentz set out to devise a time-dependent model. He derived the following set of equations

$$\frac{\partial s}{\partial x} + \frac{\partial h}{\partial t} = 0\,, \tag{5}$$

$$\frac{\partial s}{\partial t} = -\alpha \frac{\partial h}{\partial x} - \beta s|s| + \frac{1}{\rho} F\,, \tag{6}$$

where $\alpha = gq$, $\beta = gq^{-2}C^{-2}$ and F is the wind force per unit area. Further t is the time, x the coordinate measuring distance along the channel and s the flow per unit width through the channel. After proper non-dimensionalisation (we will use the same symbols for the non-dimensional counterparts), Lorentz expanded the dependent variables as follows

$$h = h_0 - h_I x + \frac{1}{2} h_{II} x^2 - \frac{1}{3} h_{III} x^3 + \cdots \tag{7}$$

and similarly for s. Substituting these expansions in Eqs. (5) and (6) and equating like powers of x, he derived a set of perturbed equations for the coefficients, which were still functions of t. Again, assuming uniform parameter conditions along each separate channel, he used these expansions to predict values at the far end of such a channel in terms of those at the initial point.

It turned out to be impracticable to carry these calculations beyond the third order. This time, only the slide rule was used. Loss of accuracy and the limitations of his slide rule prevented him from carrying on beyond that stage. He writes that the method could be used for channels up to 20 km in length. If a channel were too long, it had to be divided up into separate channels.

Lorentz came to the conclusion that the limitations of his computational aids made it impossible for him to calculate the time-dependent model for a realistic channel system such as that of Figs. 2 and 3, or even that of Fig. 4. So, he decided to do a time-dependent calculation on the simplified grid of Fig. 7, which represents the main through-flow of the system, as can be seen from Figs. 5 and 6. In this calculation the observed sea-water levels at a and g are now time-dependent and so is the efflux at v in the open situation. Carrying out calculations on the same system, but now using the stationary model, Lorentz got an idea of the small adjustments that had to be made concerning maximum water levels.

There is much more to be said about this, but that will be beyond the scope of this issue. Let it be sufficient to say that Lorentz's predictions were confirmed later on with astonishing accuracy [21] by observations made during subsequent gale conditions, that is, after the completion of the dam. This should be a sobering thought for those who see themselves confronted with impossibly complicated problems in our day and age. Had the Zuiderzee problem arisen now instead of 75 years ago, the aspiring researcher would probably have decided to write down the complete set of relevant equations in their most daunting form, *i.e.* fully three-dimensional and using the latest data, so that a full-scale picture of the bottom topography could be fed into the computer. Yet, concerning some of the physical phenomena, such as friction at the sea-bed and wind-wave interaction, such a model would still be fairly uncertain. Using large-scale parallel computing on the latest supercomputer, the present-day scientist would obtain results which, in all probability, would, at best, only be fractionally better than those obtained by Lorentz. Had he used Lorentz's approach, he would have been able to do his calculations on the simplest of personal computers and in a matter of seconds. But then, apart from a computer, he would also need the insight and modelling powers of a man such as Lorentz. There are some now who have these powers, but I feel that all too often the lure of modern computing prevents them from using these the way Lorentz did and had to do. Are we gradually losing a valuable craft?

At any rate, Lorentz's work on the Zuiderzee is everything that good engineering mathematics ought to be: 1) there is a meaningful question or problem arising outside mathematics; 2) appropriate and insightful mathematical modelling; 3) interesting mathematical analyses appropriate for the problem at hand; 4) numerical calculations which aim at bringing forth the salient features of the solution; 5) reporting the results in a manner which is geared to the needs of and can be understood by those who posed the problem in the first place and are, in general, non-mathematicians. And further, all calculations of a specialised technical nature are presented in appendices, so as not to interfere with the main line of thought. All this was admirably demonstrated in [16].

4. Lorentz: the man, the scientist, his personality, his life

Reading about H.A. Lorentz in reports written by his contemporaries, we get the impression that he must have had a remarkable personality. Apparently, he was of a mild and magnanimous disposition and free from all conceit. His colleague in Leiden, Kamerlingh Onnes - also a Nobel prize winner - once said [23, pp10,11]: "Whenever he is present, the conversation quite naturally becomes centred about him. There is nothing contradictory in this, it is self-evident and completely natural."

When his successor in Leiden, Ehrenfest, complained that he himself had not been able to solve a particular problem, Lorentz tried to console him as follows: "There is no reason to feel defeated by this, as it is clear to everyone that you have done the best you could. To do that, each one of us in his own special way and according to his talent, is really all that is required of us." Mild as it sounds, some may be inclined to read some hidden irony in this.

When a not-so-gifted student told him in despair that his briefcase had been stolen with his dissertation in it, Lorentz told him not to worry, as he would rewrite the thesis for him. The briefcase was never recovered [23, p36].

In his later years, when he had become the natural leader of the international scientific fora, Lorentz made serious attempts to keep science out of the clutches of international politics. That was not an easy task in the days after World War I, when German scientists had been banned from international meetings and were shunned almost everywhere, particularly in France and Britain. Notwithstanding his diplomatic skills and perseverance, this unfortunate situation persisted for many years.

His personal success and leadership among his contemporaries was due in part to his remarkable linguistic skills. Apart from his native language, Dutch, he appears to have been absolutely proficient in German, French and English. And indeed, he has written many papers in each of these languages. His book of 1907 [11] bears witness to this. In his preface Lorentz even writes a kind of apology for this polyglotic presentation. Even so, linguistically, each of these papers seem to be flawless. On top of this, he is also known to have corresponded in Italian [24].

Since he did not travel outside of his native country until the age of 44, it is not clear how he acquired those striking linguistic qualities, with one exception that is: French. Although he came from fairly observant protestant stock, he lost his religious beliefs early on in life. Yet, in his home town of Arnhem, he remained an avid churchgoer; but then, he would attend the local church where the services were conducted in French!

It is remarkable that his international career only took off when he was approaching or even well into his middle age. Although he had full access to the international literature and maintained an extensive correspondence, he had never met any of his foreign colleagues, many of whom he admired greatly, until he was in his mid-forties. Until then he had been a provincial scientist who had reported some of his most important work in a language which not many foreigners commanded.

Then, in 1898, pressure was brought to bear upon him to attend an international conference in Düsseldorf, his first outside The Netherlands, and his international career took off with a vengeance. Wherever he went, he was asked to preside over meetings. He made extensive lecture tours in France, Germany and the U.S.A., addressing his audiences in their native languages. He became President-for-life of the famous Solvay conferences which brought together the foremost scientists of the day. Again, it was Einstein who wrote [5,6]: "Everyone felt his superiority, no one was depressed by it." and he continues, "For me personally he meant more than all the others I have met on my life's journey."

I find it tempting to ask the following question: Would a man such as H.A. Lorentz, had he been born in the second half of the twentieth century instead of the nineteenth, have blossomed out as a scientist and internationalist as fully and as completely as the historical H.A. Lorentz has done? I daresay the answer may have to be no. Two famous lines of Wordsworth's spring to mind: "The world is too much with us... Getting and spending we lay waste our powers." Our powers of invention perhaps? Although the poet wrote these words long before the days of Lorentz and with a different purpose - communion with nature - in mind, they seem to be extremely appropriate here. Lorentz grew up in a very provincial country, where nothing out of the ordinary happened for decades on end. Outside its borders wars were waged and whole populations revolted. By contrast, in The Netherlands a political party with the name "Anti Revolutionary" became prominent! Although there were newspapers, the media were far less demanding, oppressive and glaringly present than they are now, when no one can escape their opiniated demands. Distractions were few and life seemed to have been more regular and uneventful than it is now. The young Lorentz of today has a seemingly endless choice of how to use his brain. Chances are that he will often make the wrong choices and "lay waste his

Fig. 8. Lorentz's funeral procession as it crossed the Market Place of the city of Haarlem on the 9^{th} of February 1928.

powers of invention." And I am not just thinking of the distractions that the non-scientific world puts in the way of the aspiring scientist. Who cannot picture for himself the young, or the not-so-young student, escaping from the higher tasks of thinking originally and forcing out new ideas, who whiles away long hours sitting in front of a computer screen, gazing at endless rows of data, pictures and programs?

Interestingly enough, Lorentz might have become a mathematician instead of a physicist. During his entire life, he maintained a strong propensity for and sympathetic attitude towards mathematics. In 1877, at the age of 24, he was offered a mathematics chair at the University of Utrecht. While on a hiking trip through the mountains he decided to turn it down and, instead, apply for a position as a teacher of physics at a secondary school in Leiden. One year later he was offered the newly created chair of Theoretical Physics at the University in that same city. He accepted.

We can only wonder what would have been his contribution to science had he become a mathematician instead of a physicist. With his interest in the natural world, he might even have become an applied mathematician. His later work on the Zuiderzee project strongly hints at this. He might have taken a keen interest in the new findings of Korteweg and De Vries and brought the field of solitons to fruition long before it actually did. Considering his love for the experimental side of physics, which he acquired early on in life, he might have been a G.I. Taylor *avant la lettre*. But also ... he would not have been awarded a Nobel Prize and he would, in all probability, not have risen to the heights of fame as he has done in physics.

When his life came to an end in 1928, the ceremonies surrounding his burial are indicative of how the times have changed. Apparently, in those days a scientist could be a national hero. These days, the respect that was paid to him by the general public would be bestowed only on people who have risen to prominence in the worlds of sports or show business. The streets of the city of Haarlem through which the burial procession would pass were lined with rows and rows of ordinary people who had gathered in their thousands (Fig. 8). The national telegraph services paid their respects by closing down operations for a few minutes and all over the city the national tricolor was put out in mourning. The international scientific world had gathered to pay their last respects. Representing their various countries, Ehrenfest, Rutherford, Langevin and Einstein spoke at his grave. The last of these eulogized him with the following words [6, p73]: "......I stand at the grave of the greatest and noblest man of our time."

In 1953 the centenary of Lorentz's birth was commemorated in several of the cities where he had left his personal mark. One of these was the city of Arnhem, where he was born. For reasons that I have not been able to find out, the celebrations were conducted on the 31st of January, although Lorentz was born in the summer. That same night and well into the following day the country experienced the worst hurricane in living or even recorded memory. In the South-Western part of the country the dikes, which had been neglected because of World War II and because the Government had given low priority to the maintenance of the sea defences in the early post-war period, burst in many places, causing widespread devastation. Almost 2000 people lost their lives. The economic losses were enormous. By contrast, the central and northern parts of the country experienced hardly any damaging effects. The closing dam and the other dikes, which had been constructed and reinforced basically at the direction of Lorentz's models, held out triumphantly. Later, a similar project was carried out in the South-West. The State Water-Works Department, which is responsible for these projects, is said to be still using models, the origins of which can be traced back to Lorentz's ideas.

5. About this issue

Apart from this introductory paper, the present issue consists of an English translation of Lorentz's paper of 1896 and of fifteen invited contributions. The latter appear in the order in which they were received in the Editorial Office. Although it would have been easy to solicit many more contributions – indeed, these days slow viscous flow is a very popular research topic – fifteen contributions had to be considered a maximum. The size of the special issue could not be expanded too far beyond the size of two ordinary issues combined.

Each of the invited papers separately deals with further developments of the ideas put forward by Lorentz in his 1896 paper. I shall discuss these according to the order in which these fundamental ideas appeared in Lorentz's paper.

5.1. Reciprocal theorem

Brenner and Nadim derive a reciprocal theorem for micropolar fluids. These are fluids which have an internal structure which manifests itself in spin vector fields. They derive a general integral relationship, incorporating velocity, spin, stress and couple-stress fields, relating two different micropolar flows occurring in the same domain.

Felderhof discusses a theorem for the flow near bubbles with temperature-dependent surface tension and shows that there is a simple relationship between the temperature and the velocity fields. He intimates that in some cases an electrostatic version of Lorentz's reciprocity theorem could be derived.

5.2. STOKESLET (LORENTZLET)

The flow due to a point force acting within a highly viscous fluid, the stokeslet as it is now known, plays a central role in most of the papers of this issue. The papers by Liron and Pozrikidis deal with regular infinite arrays of these singular solutions. Without special precautions, the numerical evaluation of the infinite sums is very inefficient and time-consuming. Pozrikidis shows how the numerics can be made far more efficient. Liron's paper presents a very detailed description of an analytical approach to such problems. He considers three cases to illustrate the efficiency of his procedure.

The two papers by Lighthill are concerned with the locomotion of swimming microorganisms. The author points out that these creatures propel themselves with mechanisms that are of elongated shape, *viz.* their flagella. The flow induced by such a flagellum can be described by a series of singular solutions which have their origins on the centreline of the flagellum. A strict application of Lorentz's integral theorem would require a source distribution on the entire flagellum's surface. In the first of his two contributions to this issue, Lighthill discusses the equivalence of the two formulations. Of course, in the case of elongated swimming bodies, the centreline formulation is the preferred one.

In his second paper Lighthill presents a definitive description of the locomotion of swimming microbodies such as single-celled algae, bacteria and spirochetes. When these move, their flagella assume a helical shape and this calls for a study of helical distributions of stokeslets.

The paper by Blake and Otto also considers microorganisms. Their paper is concerned with the fluid motion induced by such organisms as they attempt to feed themselves. These microorganisms produce regular lashing movements with hair-like protruberances called cilia. Such motions cause periodic vortical flows, both in the vicinity of and inside these microorganisms, enabling these to attract and filter out nutrients. To describe this phenomenon, the authors employ the concept of a 'blinking' stokeslet.

5.3. INTEGRAL-EQUATION FORMULATION (BEM)

During the past two decades, Lorentz's inverse formulation of Stokes's flow problems in terms of a set of integral equations has been used extensively in so-called boundary-element (BEM) calculations of such flows. The present collection of papers offers a number of examples in kind.

In the study by Power the problem of the interaction of a solid particle and a neutrally buoyant drop, both immersed in the same fluid, is formulated. The author derives the appropriate system of integral equations of the Fredholm type and he investigates its properties, such as uniqueness.

The paper by Toose *et al.* studies the behaviour of non-Newtonian drops immersed in an otherwise Newtonian fluid. The presence of the non-Newtonian drop requires the introduction of a domain integral in the system. Numerical examples are given, *e.g.* for a drop containing an Oldroyd-B fluid.

In the paper by Van de Vorst the Lorentz system of integral equations is used in its purest form. This author investigates the sintering of systems consisting of highly viscous particles. Since the driving force in such a system manifests itself at a particle's surface, namely through surface tension and the particle's surface curvature, Eq. (7) of Lorentz's paper applies immediately. Van de Vorst presents some interesting numerical results on the sintering of regular particle lattices.

5.4. Reflection theorem

There are five papers in the present collection dealing with this subject in one way or another.

Hasimoto's theoretical paper presents an alternative derivation of Lorentz's reflection theorem in the cases of a plane and a sphere. He then applies his theorem to three problems which were originally formulated by Lamb.

Maul and Kim also consider the reflection theorem for a spherical container and they find an alternative formulation for a solution that was originally put forward by Oseen. This new formulation has computational advantages. Applications are envisaged in suspension rheology and in biophysics.

Cox's paper investigates the force experienced by a spherical particle positioned within a vortical flow near a plane wall. Now, depending upon the characteristics of the vortex, the sphere will either become trapped by the vortex or move away from it. This indicates that, in the case of a particle suspension, such a vortical region will either end up devoid of particles or, alternatively, will be filled up with these.

Davis employs the reflection theorem to describe the flow around a disk which is moving along a plane wall. The flow induced by the disk is modelled by a distribution of stokeslets on the surface of the disk, and a set of integral equations in the manner of Lorentz ensues. Expressions for the drag and the torque experienced by the disk are derived.

The study by Keller and Ward is the only one in the set which is devoted to a study of the effects of the inertial terms when these are small, but not quite negligible. For the flow around a moving cylinder, they extend the existing series expansions, so as to include all orders. They point out that their method could be used to derive an accurate expression for the drag experienced by a small sphere moving along a plane wall in a low-Reynolds-number flow when the radius of the sphere is much smaller than its distance from the wall. In the final section of his paper Lorentz attempted to describe a method to calculate this drag. In [2, page 40] he presented the first two terms.

6. Concluding remarks

It is hoped that this special issue of the Journal of Engineering Mathematics will help to give Lorentz's work in fluid mechanics the widespread recognition that it deserves. There are some who have known all along the importance of this aspect of his work, but, as Einstein wrote, many do not realize that several of the ideas they deem important sprang from Lorentz's mind. Should the present issue achieve this, then it will have accomplished its primary goal.

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